RELATIVE HYPERBOLICITY AND ARTIN GROUPS

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ABSTRACT. This paper considers the question of relative hyperbolicity of an Artin group with regard to the geometry of its associated Deligne complex. We prove that an Artin group is weakly hyperbolic relative to its finite (or spherical) type parabolic subgroups if and only if its Deligne complex is a Gromov hyperbolic space. For a 2-dimensional Artin group the Deligne complex is Gromov hyperbolic precisely when the corresponding Davis complex is Gromov hyperbolic, that is, precisely when the underlying Coxeter group is a hyperbolic group. For Artin groups of FC type we give a sufficient condition for hyperbolicity of the Deligne complex which applies to a large class of these groups for which the underlying Coxeter group is hyperbolic.

1. INTRODUCTION

Let G denote a finitely generated group, and $\mathcal{H} = \{H_1, H_2, \dots, H_n\}$ a finite family of subgroups of G. Let Γ_S denote the Cayley graph of G with respect to a finite generating set S. We denote by $\Gamma_{S,\mathcal{H}}$ the *coned-off Cayley graph* with respect to $\mathcal{H}$. Namely, $\Gamma_{S,\mathcal{H}}$ is the graph obtained from Γ_S by introducing a vertex V_{gH} for each left coset gH, with $g \in G$ and $H \in \mathcal{H}$, and attaching V_{gH} by an edge of length $\frac{1}{2}$ to each vertex of Γ_S labelled by an element of the coset gH. The isometric (left) action of G on the Cayley graph Γ_S clearly extends to an isometric action of G on $\Gamma_{S,\mathcal{H}}$ (by setting $g(V_{g'H}) = V_{gg'H}$ for all $g, g' \in G$ and $H \in \mathcal{H}$). The nontrivial vertex stabilizers of this action are all conjugate to subgroups of $\mathcal{H}$.

The group G is said to be *weakly hyperbolic relative to $\mathcal{H}$* if the coned-off Cayley graph $\Gamma_{S,\mathcal{H}}$ is a Gromov hyperbolic space. This definition can be shown to be independent of the choice of finite generating set S (see [9]). (We also refer the reader to [10, 11] or other standard references, such as [5], for the definition of a Gromov hyperbolic space).

The notion of weak relative hyperbolicity just described was first studied by B. Farb in [9]. We remark that the more conventional notion of relative hyperbolicity introduced by M. Gromov [10, 11], and explored by B. Bowditch [4], A. Yaman [16] and others is rather more specific. It is equivalent to weak relative hyperbolicity as defined above together with the additional condition of *bounded coset penetration* (BCP) introduced by Farb in [9]. The stronger form of relative hyperbolicity à la Gromov models more precisely certain classical situations such as that of a geometrically finite Kleinian group (which is hyperbolic relative to its parabolic subgroups in the stronger sense).

The difference between these two notions of relative hyperbolicity is perfectly and rather starkly illustrated in the case of Artin groups (defined in Section 2 below). In [12], I. Kapovich and P. Schupp showed that Artin groups with all relator indices at least 7 are weakly hyperbolic relative to their non-free rank 2 parabolic subgroups (in this case the finite type parabolics). On the other

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hand, as observed in [12], the presence of numerous mutually intersecting free abelian subgroups tends to rule out altogether the possibility of relative hyperbolicity in the stronger sense for most Artin groups. More precisely, it is shown in [2] and also follows easily from Lemma 4 of [1] that if an Artin group G is strongly relatively hyperbolic with respect to a family of groups $\mathcal{H}$ then each freely indecomposable free factor of G must be included in $\mathcal{H}$. In particular, a freely indecomposable Artin group is strongly relatively hyperbolic in only the most trivial of senses, namely with respect to itself.

In this paper we shall focus on *weak* relative hyperbolicity and, following [12], we shall consider hyperbolicity of an Artin group relative to its finite type parabolic subgroups in this sense. By considering this question in connection with the geometry of the associated Deligne complex (see Section 2) we are able to extend the results of [12] considerably while giving a more unified approach to the problem. Thus we have:

Theorem 1.1. *An Artin group is weakly hyperbolic relative to its finite type standard parabolic subgroups if and only if its Deligne complex is a Gromov hyperbolic space.*

In Section 3 we recall the definition of a *2-dimensional* Artin group. These include the groups considered in [12]. It is an interesting fact that the Deligne complex for a 2-dimensional Artin group is Gromov hyperbolic if and only if the Davis complex associated with the underlying Coxeter group is. We therefore obtain the following (see Proposition 3.1).

Theorem 1.2. *A 2-dimensional Artin group is weakly hyperbolic relative to its finite type standard parabolic subgroups if and only if the corresponding Coxeter group is a hyperbolic group.*

In the absence of a counter-example, it is reasonable to conjecture that the statement of Theorem 1.2 holds for an arbitrary Artin group.

Conjecture 1.3. *An Artin group is weakly hyperbolic relative to its finite type standard parabolic subgroups if and only if the corresponding Coxeter group is a hyperbolic group.*

In view of Theorem 1.1 the conjecture is equivalent to stating that, for any Artin group, the Deligne complex is Gromov hyperbolic if and only if the Davis complex associated to the corresponding Coxeter group is Gromov hyperbolic. In one direction, this conjecture is easy to prove. In Section 3, we give a simple proof that hyperbolicity of the Deligne complex implies hyperbolicity of the Davis complex. On the other hand, we are currently a long way from proving the converse. The global geometry of the Deligne complex is not well understood in general. It is not known, for example, whether every Deligne complex supports an equivariant metric of nonpositive curvature (a CAT(0) metric). Even in cases where such a metric known, the question of the existence of flat planes (and hence of Gromov hyperbolicity) is still rather delicate.

Apart from the 2-dimensional case, the main situation where the Deligne complex is known to admit a CAT(0) metric is in the case of an Artin group of FC type. In this case the preferred metric is cubical. In Section 4, we describe a systematic method of deforming the metric in each cube of the Deligne complex so as to obtain a piecewise hyperbolic metric. We than obtain a sufficient condition for this deformed metric to be CAT(-1). The condition is simply that the defining graph (as defined in Section 2) for the FC type Artin group has *no empty squares*, meaning that any circuit of length four has at least one diagonal pair of vertices spanning an edge. (We remark that "no empty squares" in the defining graph does not imply "no empty squares" in the link of every simplex of the Deligne complex, hence this deformation must be done carefully.) As a consequence we have the following.

Theorem 1.4. *An FC type Artin group whose defining graph has no empty squares is weakly hyperbolic relative to its finite type standard parabolic subgroups.*

Note that while all the Artin groups covered by Theorem 1.4 are necessarily associated with hyperbolic Coxeter groups, the "no empty squares" condition does not quite capture every hyperbolic Coxeter group of FC type. However, in this case, we are actually proving somewhat more than is required for Theorem 1.4, namely we show that the Deligne complex is CAT(-1) (which is a priori stronger than Gromov hyperbolic), and this for a rather specific choice of metric.

We conclude this introduction with a remark on the proof of Theorem 1.1. The key observation is that the action of the Artin group on its Deligne complex is particularly well-adapted to studying the group in relation to its finite type parabolic subgroups simply because these subgroups are precisely the isotropy subgroups of the action. One can therefore relate the geometry of the coned-off Cayley graph to that of the Deligne complex (via a quasi-isometry) in order to prove Theorem 1.1. As it happens, the most natural way of expressing this argument is in the form of a more general "relative" Milnor-Svarc Lemma. Recall that the usual Milnor-Svarc Lemma states that if a finitely generated group G acts properly discontinuously, cocompactly and isometrically on a length space X, then G and X are quasi-isometric spaces. In particular, G is a hyperbolic group if and only if X is Gromov hyperbolic. The relative version states that if a finitely generated group G acts *discontinuously* (i.e. with discrete orbits), cocompactly and isometrically on a length space X then X is quasi-isometric to the coned-off Cayley graph $\Gamma_{S,\mathcal{H}}$ for G, where S is any finite generating set and $\mathcal{H}$ denotes the collection of maximal isotropy subgroups. We defer the proof (and a careful statement) of this result to Section 5, Theorem 5.1, even though we shall use it almost immediately in Section 2, below.

2. Relative hyperbolicity and Artin groups

Let Δ denote a simplicial graph with vertex set $V(\Delta)$ and edge set $E(\Delta) \subset V(\Delta) \times V(\Delta)$. Suppose also that every edge $e = \{s, t\} \in E(\Delta)$ carries a label $m_e = m_{st} \in \mathbb{N}_{\geq 2}$. We define the *Artin group* $G(\Delta)$ associated to the (labelled) *defining graph* Δ to be the group given by the presentation [1]

$$G(\Delta) = \langle\ V(\Delta)\ \mid\ \underbrace{ststs\cdots}_{m_{st}} = \underbrace{tstst\cdots}_{m_{st}}\ \text{ for all } \{s, t\} \in E(\Delta)\ \rangle.$$

Adding the relations $s^2 = 1$ for each $s \in V(\Delta)$ yields a presentation of the associated *Coxeter group* $W(\Delta)$ of type Δ. We denote $\rho_\Delta : G(\Delta) \to W(\Delta)$ the canonical quotient map obtained by this addition of relations. An Artin group is said to be of *finite type* (sometimes written *spherical type*) if the associated Coxeter group is finite, and of *infinite type* otherwise. By a *standard parabolic subgroup* of $G(\Delta)$, or $W(\Delta)$, we mean any subgroup generated by a (possibly empty) subset of the standard generating set $V(\Delta)$. More generally, any subgroup which is conjugate to a standard parabolic subgroup (of $G(\Delta)$ or $W(\Delta)$) shall be referred to as a *parabolic subgroup*.

Probably the most important tool currently used in the study of *infinite* type Artin groups is the Deligne complex (see [7], etc..). We described this complex in detail.

Consider $G = G(\Delta)$ for a fixed defining graph Δ. For each subset of the generating set, $R \subset V(\Delta)$, we shall write Δ_R for the full labelled subgraph of Δ spanned by R. (Here we attach a meaning to the *empty* defining graph $\Delta_\emptyset$ by setting $G(\Delta_\emptyset) = 1$ and $W(\Delta_\emptyset) = 1$). The inclusion of

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[1]Our notion of defining graph differs from the frequently used "Coxeter graph" where, by contrast, the absence of an edge between s and t indicates a commuting relation ($m_{st} = 2$) and the label $m_{st} = \infty$ is used to designate the absence of a relation between s and t. In our convention the label ∞ is never used.

Δ_R in Δ induces a homomorphism $\phi_R : G(\Delta_R) \to G$ with image the standard parabolic subgroup $\langle R \rangle$ generated by R. The construction of the Deligne complex is based on the rather nontrivial fact, due to H. van der Lek [13], that, for every defining graph Δ and every $R \subset V(\Delta)$, the homomorphism ϕ_R is an isomorphism onto its image. Thus each standard parabolic subgroup of an Artin group is itself canonically isomorphic to an Artin group. The corresponding statement for Coxeter groups is also true, and well-known (see [3]).

Define
$$\mathcal{V}_f \;=\; \{R \subset V(\Delta) \,:\, W(\Delta_R) \text{ finite}\,\}.$$
We view $\mathcal{V}_f$ as a partially ordered set under inclusion of sets, and define K to be the geometric realisation of the derived complex of $\mathcal{V}_f$. Thus there is a simplex $\sigma \in K$ of dimension $n \geq 0$ for every chain $R_0 \subset R_1 \subset \cdots \subset R_n$ of $n+1$ distinct elements in $\mathcal{V}_f$. We denote $min(\sigma) = R_0$, the minimal vertex of σ.

Note that, for $\emptyset \subseteq R \subset T \subseteq V(\Delta)$, the inclusion $\Delta_R \subset \Delta_T$ induces a homomorphism $\phi_{R,T} : G(\Delta_R) \to G(\Delta_T)$. It follows that setting $G(\sigma) = G(\Delta_{min(\sigma)})$ and $\phi_{\sigma,\tau} = \phi_{min(\sigma),min(\tau)}$, for all $\tau \subset \sigma \in K$, defines a *simple complex of groups* structure
$$(K, \{G(\cdot)\}, \{\phi_{\cdot,\cdot}\})$$
in the sense of [5]. It is easily seen that the (orbifold) *fundamental group* of this complex of groups is isomorphic to the Artin group G (via the homomorphisms $\phi_R : G(\Delta_R) \to G$), and the result of van der Lek cited above ensures that the complex of groups is *developable*. It follows that the Artin group G acts, with quotient K and isotropy subgroups the finite type parabolics $\{\, G(\sigma) \,:\, \sigma \in K\,\}$, on a simply connected simplicial complex, the *universal cover* of $(K, \{G(\cdot)\}, \{\phi_{\cdot,\cdot}\})$, which we shall denote $\mathbb{D}$ and refer to as the *Deligne complex* associated to $G(\Delta)$.

We note that, replacing the collection of groups $\{\, G(\sigma) \,:\, \sigma \in K\,\}$ with the corresponding Coxeter groups $\{\, W(\sigma) \,:\, \sigma \in K\,\}$ leads in a similar way to the definition of a developable complex of groups whose fundamental group is, this time, the Coxeter group $W = W(\Delta)$, and whose universal cover, which we shall denote $\mathbb{D}_W$, is known as the *Davis complex*. The Coxeter group acts on its Davis complex with finite vertex stabilizers (in fact properly discontinuously and cocompactly). This is quite different from the situation of the Deligne complex where the Artin group acts with every nontrivial vertex stabilizer an infinite group. In particular, the Deligne complex is not even a locally compact space (while the Davis complex clearly is). Nevertheless, a lot of important information is carried by the action of the Artin group on its Deligne complex, as can be seen from [7, 8] etc.

We suppose now that the complex K is endowed with a piecewise Euclidean or piecewise hyperbolic metric. Since K is finite, this induces a complete G-equivariant length metric on the Deligne complex (c.f. [5]). There are two very natural choices for such a metric (namely the Moussong metric and the cubical metric) described in [7]. These specific metrics are particularly useful when they can be shown to be nonpositively curved, or CAT(0), as demonstrated in [7], where we refer the reader for further details. However, in what follows, the actual choice of piecewise Euclidean or hyperbolic metric is more or less irrelevant since for any such metric d, the Deligne complex $(\mathbb{D}, d)$ will be (G-equivariantly) quasi-isometric to the 1-skeleton of $\mathbb{D}$ equipped with the unit-length edge metric. The statement "the Deligne complex is Gromov hyperbolic" shall henceforth be interpreted to mean "with respect to any equivariant piecewise Euclidean or piecewise hyperbolic metric".

We are now able to prove Theorem 1.1 which we restate below:

Theorem 2.1. *An Artin group is weakly hyperbolic relative to its finite type standard parabolic subgroups if and only if its Deligne complex is a Gromov hyperbolic space.*

Proof. It is easily seen that the action of an Artin group on its Deligne complex (equipped with a piecewise Euclidean length metric) is discontinuous and co-compact. (As observed later, in Remark

??, this is actually true in the case of an arbitrary finite developable complex of groups). Also, the (maximal) isotropy subgroups of this action are, by construction, just the (maximal) finite type parabolic subgroups of the Artin group. The result now follows immediately from the relative version of the Milnor-Svarc Lemma (Theorem 5.1) proved in Section 5. $\square$

3. Two-dimensional Artin groups

We say that the Artin group $G = G(\Delta)$ is *2-dimensional* if Δ has at least one edge (G is not free) and every triangle in Δ has edge labels m, n, p satisfying $1/m + 1/n + 1/p \leq 1$, equivalently if every rank 3 parabolic subgroup is of infinite type. The terminology is justified by the fact that an Artin group is 2-dimensional in this sense if and only if it has cohomological dimension 2. Each 2-dimensional Artin group is also known to have geometric dimension 2. Moreover, the Deligne complex is 2-dimensional and is CAT(0) when equipped with the Moussong metric. These statements were all established in the paper of the first author and M. Davis [7].

In [12], I. Kapovich and P. Schupp showed that an Artin group with all indices $m_{ij} \geq 7$ is weakly hyperbolic relative to its maximal finite type parabolic subgroups. It is clear that the groups treated by Kapovich and Schupp are all examples of 2-dimensional Artin groups. The following statement generalises their result.

Proposition 3.1. *Let $G(\Delta)$ be a 2-dimensional Artin group. Then the following are equivalent*

1. *the Davis complex $\mathbb{D}_W$ is Gromov hyperbolic;*
2. *the Coxeter group $W(\Delta)$ is a hyperbolic group;*
3. *Δ contains no triangle having edge labels m, n, p with $1/m + 1/n + 1/p = 1$ and no square with all edge labels equal to 2.*
4. *the Deligne complex $\mathbb{D}$ (equipped with the Moussong metric) is Gromov hyperbolic;*
5. *$G(\Delta)$ is weakly hyperbolic relative to the collection of finite type standard parabolic subgroups, namely the set $\{G(e) : e \in E(\Delta)\}$.*

Proof. The equivalence of (1) and (2) follows by an application of the Milnor-Svarc Lemma and the equivalence of these and (3) follows immediately from Moussong's conditions [15] (see discussion below). The equivalence of (1-3) and (4) is proved in [8] Lemma 5 (using the Flat Plane Theorem and the relationship between the Deligne complex and the Davis complex). Finally, the equivalence of (4) and (5) is a consequence of the above Theorem 2.1. $\square$

We observe that there is quite generally a close relationship between the Deligne complex $\mathbb{D}$ associated to an Artin group $G = G(\Delta)$ and the Davis complex $\mathbb{D}_W$ for the corresponding Coxeter group $W = W(\Delta)$. The canonical projection induces a simplicial map $\mathbb{D} \to \mathbb{D}_W$ which is surjective, and the Tits section (a setwise section to the canonical projection) induces an inclusion $\mathbb{D}_W \hookrightarrow \mathbb{D}$. We suppose that $\mathbb{D}_W$ is equipped with an equivariant metric. Since the Coxeter group W acts properly co-compactly and isometrically on $\mathbb{D}_W$ the space and the group are quasi-isometric (by the Milnor-Svarc Lemma). Thus W is a hyperbolic group precisely when $\mathbb{D}_W$ is a Gromov hyperbolic space.

By the work of Moussong [15], the Davis complex equipped with the Moussong metric $(\mathbb{D}_W, d_M)$ is, in all cases, a CAT(0) space and is Gromov hyperbolic if and only if it contains no isometrically embedded flat plane. Moreover, Moussong shows that $\mathbb{D}_W$ contains an embedded flat only when W contains an obvious $\mathbb{Z} \times \mathbb{Z}$ subgroup, namely when W contains a standard parabolic subgroup which is either Euclidean or is the direct product of two infinite standard parabolics.

In summary, an arbitrary Coxeter group W is hyperbolic group if and only if both of the following conditions hold:

(M1) no standard parabolic subgroup of W is a Euclidean reflection group; and,

(M2) no standard parabolic subgroup of W is the direct product of two infinite parabolics.

An analogous metric, also known as the Moussong metric, can be defined on the Deligne complex $\mathbb{D}$ such that the inclusion and projection maps $\mathbb{D}_W \to \mathbb{D} \to \mathbb{D}_W$ are isometries on each simplex. It is conjectured in [7] that the Moussong metric on $\mathbb{D}$ is always CAT(0). This conjecture is only known to hold in the 2-dimensional case and in some limited higher dimensional cases containing four-strand braid groups as parabolics (see [6]). We can use this metric, however, to prove the easy direction of Conjecture 1.3.

Proposition 3.2. *With respect to the Moussong metric, the inclusion of $\mathbb{D}_W \hookrightarrow \mathbb{D}$ induced by the Tits section is an isometric embedding. In particular, if $\mathbb{D}$ is Gromov hyperbolic, then so is $\mathbb{D}_W$.*

Proof. Since the natural projection $\mathbb{D} \to \mathbb{D}_W$ and the inclusion $\mathbb{D}_W \to \mathbb{D}$ take simplices isometrically onto simplices, they map geodesics to piecewise geodesics. It follows that both maps are distance non-increasing. But the composite map $\mathbb{D}_W \to \mathbb{D} \to \mathbb{D}_W$ is an isometry (in fact it is the identity map), so the first of these maps must also be distance non-decreasing, i.e., it is an isometric embedding. □

4. FC TYPE ARTIN GROUPS

The only other case in which the Deligne complex has been shown to have a CAT(0) metric is the case of an Artin group of FC type. A defining graph Δ, and its associated Artin group $G(\Delta)$, is said to be of *FC type* if it satisfies the following condition:

$T \subset V(\Delta)$ lies in $\mathcal{V}_f$ if and only if T spans a complete subgraph in Δ.

(Recall that a graph is complete if any two vertices are connected by an edge.) There is a natural cubical structure on the Deligne complex of any Artin group which we will describe below. The FC condition precisely guarantees that all links in this cubical structure are flag complexes (hence the terminology "FC") and so the induced geodesic metric on $\mathbb{D}$ is CAT(0).

For FC groups, there is a corresponding CAT(0) cubical structure on the Davis complex and the embedding $\mathbb{D}_W \hookrightarrow \mathbb{D}$ is isometric. It is thus clear that Moussong's conditions for hyperbolicity of the Davis complex are necessary conditions for hyperbolicity of the Deligne complex. We will prove a partial converse of this statement. We say that a graph Δ has *no empty squares* if for any circuit of length four, at least one diagonal pair of vertices spans an edge. We will show that if the defining graph for an Artin group is FC type and has no empty squares, then the Deligne complex $\mathbb{D}$ supports a hyperbolic metric. Note that these hypotheses are strictly stronger than Moussong's. The FC condition rules out Euclidean parabolics in W, and the second condition rules out products of infinite parabolics (since by the FC condition, any infinite parabolic contains a pair of generators not connected by an edge in Δ). On the other hand, the graph below with all edges labelled 5 gives a defining graph satisfying both of Moussong's conditions (M1-M2), but neither of the two conditions above.

In the case of a right-angled Artin group, that is, one in which all edges in the defining graph are labeled 2, the no empty squares condition exactly agrees with Moussong's conditions, thus Conjecture 1.3 holds for these groups.

The idea behind the proof of hyperbolicity is to slightly deform the CAT(0) cubical metric to make it negatively curved. The no empty squares condition guarantees that some (but not all) of the links in this metric are "extra large", that is, the lengths of closed geodesics are bounded away from 2π, hence the metric at those vertices can be deformed slightly without destroying the
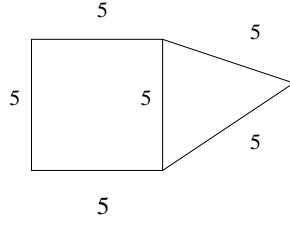
FIGURE 1. Defining graph Γ

CAT(0) condition. The trick is to replace the Euclidean cubes with hyperbolic cubes which contain enough right angles to preserve the metric on those links which are not extra large.

Define a hyperbolic metric on an n-cube as follows. Identify n-dimensional hyperbolic space with the hyperboloid

$$\mathbb{H}^n = \{(x_1, \ldots, x_n, x_{n+1}) \mid x_1^2 + \cdots + x_{n-1}^2 - x_n^2 = -1, x_n > 0\}$$

in $\mathbb{R}^{n,1}$. Let $H \cong (\mathbb{Z}/2)^n$ be the reflection group on $\mathbb{H}^n$ generated by reflections across the coordinate hyperplanes $x_i = 0, i = 1, \ldots, n$. For any $\epsilon > 0$, let x_ϵ be the (unique) point in the positive orthant of $\mathbb{H}^n$ at distance ϵ from every coordinate hyperplane. Let Y_ϵ^n be the convex hull in $\mathbb{H}^n$ of the H-orbit of x_ϵ. Then Y_ϵ^n is a regular hyperbolic n-cube of side length 2ϵ. Clearly, any k-dimensional face of Y_ϵ^n is isometric to Y_ϵ^k.

Now let C_ϵ^n be the intersection of Y_ϵ with the positive orthant. Then C_ϵ^n is again a hyperbolic n-cube, but it is not regular. The vertex $x_0 = (0, \ldots, 0, 1)$ has all codimension one faces meeting at right angles whereas the vertex x_ϵ has all codimension one faces meeting at some angle $\theta < \frac{\pi}{2}$ that depends on ϵ. As ϵ goes to zero, the metric on Y_ϵ approaches the Euclidean metric, so we can take θ arbitrarily close to $\frac{\pi}{2}$ by choosing ϵ sufficiently small.

Define the *type* of a vertex v in C_ϵ^n to be the number of coordinate hyperplanes containing v. Thus, x_0 has type n while x_ϵ has type 0. A face F of C_ϵ^n contains a unique vertex of minimal type k, and a unique vertex of maximal type l. We say the face F has *type* (k, l). By symmetry, it is easy to see that all faces of a given type (k, l) in C_ϵ^n are isometric. The next lemma shows that the metric on a face of type (k, l) is independent of n.

Lemma 4.1. *Suppose $F \subseteq C_\epsilon^n$ is of type (k, l). Then F is isometric to some (hence any) face of type (k, l) in C_ϵ^l. In particular, a face of type $(0, l)$ is isometric to C_ϵ^l.*

Proof. First consider a face F of type $(0, l)$. Any such face is the intersection of a l-dimensional face $\hat{F}$ of Y_ϵ^n with the positive orthant. Since $\hat{F}$ is isometric to Y_ϵ^l, we conclude that F is isometric to C_ϵ^l. More generally, any face F of type (k, l) is contained in a face of type $(0, l)$, so F is isometric to a face of type (k, l) in C_ϵ^l. $\square$

Next, we analyze links of vertices in the cube C_ϵ^n. By construction, the link of x_0 is a spherical simplex with all edge lengths $\pi/2$ (called an "all-right spherical simplex"), while the link of x_ϵ is a regular spherical simplex with all edge lengths $\theta < \frac{\pi}{2}$.

Suppose v is a vertex of type k, $0 < k < n$. Every codimension 1 face of C_ϵ^n containing v either lies in a coordinate hyperplane, or in a codimension 1 face of Y_ϵ^n. Since a codimension 1 face of Y_ϵ^n is preserved by reflection across every coordinate hyperplane it intersects, it follows that these two types of faces intersect orthogonally. Thus, the link of v decomposes as an orthogonal join,

$$link(v, C_\epsilon^n) = link_\uparrow(v, C_\epsilon^n) * link_\downarrow(v, C_\epsilon^n)$$

where $Link_\uparrow(v, C_\epsilon^n)$ is the link of v in the face F_0 spanned by v and x_0, while $Link_\downarrow(v, C_\epsilon^n)$ is the link of v in the face F_ϵ spanned by v and x_ϵ. The face F_0 is of type (k, n) while F_ϵ is of type $(0, k)$. By the lemma above, there is an isometry of F_ϵ with C_ϵ^k taking v to the basepoint x_0 (in C_ϵ^k). Thus, the downward link is an all-right simplex.

We now define a piecewise hyperbolic metric on the Deligne complex $\mathbb{D}$. For this we will use a slightly different description of $\mathbb{D}$. Let $G = G(\Delta)$ and for $T \in \mathcal{V}_f$, write $G_T = G(\Delta_T)$. A fundamental domain for the action of G on $\mathbb{D}$ is the complex K defined in the previous section. If we think of the vertices of K as finite type parabolic subgroups G_T rather than as sets of generators $T \in \mathcal{V}_f$, then the vertices in $\mathbb{D}$ correspond to cosets gG_T, $T \in \mathcal{V}_f$ and simplices to totally ordered flags of cosets.

This gives rise to a cubical structure on $\mathbb{D}$ whose vertices are the cosets gG_T and whose cubes correspond to "intervals",

$$[gG_T, gG_R] = \text{span of the vertices } gG_{T'} \text{ with } gG_T \subseteq gG_{T'} \subseteq gG_R.$$

Define an equivariant, piecewise hyperbolic metric d_ϵ on $\mathbb{D}$ by assigning each cube $[gG_T, gG_R]$ with $|T| = t$, $|R| = r$ the metric of a face of type (t, r) in C_ϵ^r. In particular, the cubes $[G_\emptyset, G_R]$ are isometric to C_ϵ^r with the vertex $G_\emptyset$ identified with x_ϵ and G_R identified with x_0.

Theorem 4.2. *Suppose an Artin group $G(\Delta)$ is FC type, and its defining graph has no empty squares. Then the metric d_ϵ on the Deligne complex $\mathbb{D}$ is CAT(-1) for ϵ sufficiently small. In particular, $\mathbb{D}$ is Gromov hyperbolic.*

Proof. First consider the simplicial complex L whose vertex set is $V(\Delta)$ and whose simplices σ_T are spanned by the subsets $T \in \mathcal{V}_f$. (This is known as the "link" of the Coxeter group $W(\Delta)$.) The hypotheses of the theorem precisely guarantee that this simplicial complex satisfies Seibenman's "no triangles, no squares" condition. Moreover, the FC condition implies that the link of any simplex in L is isomorphic (as a simplicial complex) to a full subcomplex of L. Hence these links also satisfy the "no triangles, no squares" condition. In [10], Gromov shows that if each simplex in such a complex is an all-right spherical simplex, then the resulting geodesic metric is "extra large". That is, L is CAT(1) and closed geodesics in L, and in all links in L, have lengths bounded away from 2π. As shown by Moussong in [15], any sufficiently small deformation of the metric on each simplex preserves this property, hence the resulting metric remains CAT(1).

To prove the theorem, we must show that the link of every vertex v in $\mathbb{D}$ is CAT(1) with respect to the metric d_ϵ. By equivariance of the metric, it suffices to consider vertices $v = G_T$ lying in the fundamental domain K. By the discussion above, the link of v in $\mathbb{D}$ decomposes as an orthogonal join

$$link(v, \mathbb{D}) = link_\uparrow(v, \mathbb{D}) * link_\downarrow(v, \mathbb{D}),$$

so it suffices to show that the upward and downward links are each CAT(1).

The upward link consists of

$$link_\uparrow(v, \mathbb{D}) = \bigcup_{T \subset R} link(v, [G_T, G_R]).$$

It has a k-simplex for each spherical parabolic $G_R \supset G_T$ with $|T| - |R| = k - 1$. Thus, as an abstract simplicial complex, it can be identified with the complex L defined above when $T = \emptyset$, and with $link(\sigma_T, L)$ otherwise. The metric on $link_\uparrow(v, \mathbb{D})$ induced by d_ϵ is a peicewise spherical metric with all edge lengths arbitrarily close to $\frac{\pi}{2}$ for sufficiently small ϵ. It follows from the discussion above that this metric is CAT(1).

The downward link, $link_\downarrow(v, \mathbb{D})$ is composed of the link of G_T in the cubes $[gG_\emptyset, G_T]$, for $g \in G_T$. It is an all-right piecewise spherical complex which is isomorphic to the link of the maximal vertex G_T in the Deligne complex for the finite type Artin group G_T. This was shown to be a flag complex in [7], Lemma 4.3.2, hence it is CAT(1). $\qquad\square$

Corollary 4.3. *An Artin group of FC type whose defining graph has no empty squares is hyperbolic relative to its finite type standard parabolic subgroups.*

5. A RELATIVE VERSION OF THE MILNOR-SVARC LEMMA

Let X be a metric space and G a group which acts on X by isometries. We say that the action is *co-compact* if there exists a compact set $K \subset X$ such that $X = \bigcup_{g \in G} gK$, and *discontinuous* if every orbit is a discrete subspace of X, equivalently if for all $x \in X$ there exists an $\epsilon_x > 0$ such that $d(x, y) > \epsilon_x$ for all $y \in G(x) \setminus \{x\}$. A subgroup $H < G$ is said to be an *isotropy subgroup* of G if its fixed set in X is non-empty.

Theorem 5.1. *Let G be a finitely generated group and suppose that G admits a discontinuous, co-compact, isometric action on a length space X. Let $\mathcal{H}$ denote a collection of subgroups of G consisting of exactly one representative of each conjugacy class of maximal isotropy subgroups for the action of G on X. Then $\mathcal{H}$ is finite and, for any finite generating set S of G, the coned-off Cayley graph $\Gamma_{S,\mathcal{H}}(G)$ is quasi-isometric to X. In particular, if X is a Gromov hyperbolic space then the group G is weakly hyperbolic relative to the collection $\mathcal{H}$ of maximal isotropy subgroups.*

Proof. We first show that the number of conjugacy classes of maximal isotropy subgroups is finite. Clearly, distinct maximal isotropy subgroups have disjoint fixed sets in X. Also, since G acts co-compactly, every maximal isotropy subgroup is conjugate to one which fixes a point inside a certain compact region K such that $X = \bigcup_{g \in G} gK$. By way of contradiction, we now suppose that there exists an infinite sequence $H_1, H_2, \ldots$ of pairwise distinct maximal isotropy subgroups such that each H_i fixes a point $x_i \in K$ (the points x_i being necessarily pairwise distinct). By compactness of K we may pass to an infinite subsequence for which the sequence $(x_i)_{i \in \mathbb{N}}$ converges to a point $x \in K$. Moreover, at most one of the H_i may fix x, so we may as well suppose that none of them fix x. For each i we may therefore choose an element $h_i \in H_i$ such that $h_i(x) \neq x$. Since $d(h_i(x), x) \leq 2d(x_i, x)$ it follows that the sequence $(h_i(x))_{i \in \mathbb{N}}$ also converges to x, contradicting the assumption that G acts discontinuously.

Let $H_1, H_2, \ldots, H_n$ denote the finitely many maximal isotropy subgroups whose fixed sets intersect K nontrivially. Set $Q = \{g \in G : gK \cap K \neq \emptyset$ but $g \notin H_r$ for all $r = 1, \ldots, n\}$, and choose a subset $\widehat{Q} \subset Q$ which contains exactly one representative for each coset gH_r, for $g \in Q$ and $r \in \{1, \ldots, n\}$. We note that G is generated by the set $\widehat{Q}$ together with the subgroups $H_1, \ldots, H_n$.

We now use a compactness argument to show that $\widehat{Q}$ is finite. By way of contradiction we suppose that there exists an infinite sequence $(g_i)_{i \in \mathbb{N}}$ of pairwise distinct elements of $\widehat{Q}$. For each $i \in \mathbb{N}$ we may find $x_i, y_i \in K$ such that $g_i(x_i) = y_i \in gK \cap K$. Since $K \times K$ is compact we may pass to an infinite subsequence of $(g_i)_{i \in \mathbb{N}}$ for which the sequence of pairs (x_i, y_i) converges to a point $(x, y) \in K \times K$. Since

$$d(g_i(x), y) \leq d(g_i(x), g_i(x_i)) + d(g_i(x_i), y) = d(x, x_i) + d(y_i, y)$$

it follows that the sequence $(g_i(x))_{i \in \mathbb{N}}$ converges to y. Since the action of G is discontinuous, this implies that the sequence is eventually constant: there exists N such that $g_i(x) = y$ for all $i > N$. But then, for any $N < i < j$, the element $g_j^{-1}g_i$ fixes $x \in K$ and hence lies in some maximal

isotropy subgroup H_r. That is to say that g_i and g_j are different representatives in $\widehat{Q}$ for the same coset of H_r, contradicting the choice of $\widehat{Q}$.

Since G is finitely generated we may extract a finite generating set from any given generating set for the group. It follows that we may extend $\widehat{Q}$ to a finite generating set S of G in such a way that $\widehat{Q} \subset S \subset \widehat{Q} \cup H_1 \cup \cdots \cup H_n$. Let $\Gamma = \Gamma_{S,\mathcal{H}}(G)$ denote the coned-off Cayley graph for G with respect to the generating set S and a finite set $\mathcal{H}$ of isotropy subgroups as prescribed in the statement of the Theorem. We may as well suppose that $\mathcal{H}$ is chosen by selecting from amongst the subgroups H_r, $r = 1, \ldots, n$, one from each conjugacy class. Moreover, the exact choice of representatives for $\mathcal{H}$ is not really important, as the structure of the coned-off Cayley graph Γ depends only on the set of conjugacy classes of subgroups involved. Note also that, up to quasi-isometry, Γ depends only on the set $\widehat{Q}$ (rather than the choice of S) and the subgroups H_r. More generally, the coned-off Cayley graph $\Gamma = \Gamma_{S,\mathcal{H}}(G)$ is independent, up to quasi-isometry, of the choice of generating set S (regardless of whether or not it contains $\widehat{Q}$) as long as this set is finite.

Let v_0 denote the base vertex of Γ and write Γ_0 for the G-orbit of v_0 with metric induced from Γ. Then the inclusion $\Gamma_0 \to \Gamma$ is a quasi-isometry. We shall show that Γ_0 (and therefore Γ) is quasi-isometric to X.

Choose a point $x_0 \in K$. This choice determines a G-equivariant map $f : \Gamma_0 \to X$ by sending gv_0 to gx_0 for all $g \in G$. It is easily seen that, for $p, q \in \Gamma_0$,

$$d_X(f(p), f(q)) \leq R\, d_\Gamma(p, q)\,,$$

where R denotes the maximum value in the finite set

$$\{\, d_X(x_0, s(x_0)) \;:\; s \in \widehat{Q}\,\} \cup \{\, 2d_X(x_0, \mathrm{Fix}(H_r)) \;:\; r = 1, \ldots, n\,\}\,.$$

In order to prove the reverse inequality (i.e. to bound $d_\Gamma(p, q)$ above by a linear function of $d_X(f(p), f(q))$) we need to establish the following fact:

There exists a constant $\epsilon > 0$ such that, for all $g \in G$, either $gK \cap K \neq \emptyset$ or $d_X(gK, K) > \epsilon$ (where here we understand the Hausdorff distance).

We use, once again, a compactness argument to prove this statement. If the statement is not true then we may find a sequence $(g_i)_{i \in \mathbb{N}}$ of distinct group elements such that $0 < d_X(g_{i+1}K, K) < \frac{1}{2}d_X(g_iK, K)$ for all $i \in \mathbb{N}$. Choosing, for each i, a pair $(x_i, y_i) \in K \times K$ such that $d_X(g_iK, K) < d_X(g_i(x_i), y_i) < 2d_X(g_iK, K)$, and passing to an infinite subsequence for which the sequence of pairs converges to a pair $(x, y) \in K \times K$, we observe that the sequence $(g_i(x))_{i \in \mathbb{N}}$ converges to y. This contradicts the assumption that the action of G is discontinuous unless the sequence is eventually stationary, that is, unless $g_i(x) = y$ for some i. But this is impossible since $g_iK \cap K = \emptyset$ for all $i \in \mathbb{N}$.

We shall now give an upper bound for $d_\Gamma(p, q)$. Since X is a path metric space we may find a path γ from $f(p)$ to $f(q)$ in X whose length approximates the distance between these points to within ϵ: $\ell(\gamma) \leq d_X(f(p), f(q)) + \epsilon$. Choose $m \in \mathbb{N}$ such that $(m - 1)\epsilon < \ell(\gamma) \leq m\epsilon$, and let $f(p) = y_0, y_1, \ldots, y_m = f(q)$ denote equally spaced points along the path γ. In particular $d(y_{i-1}, y_i) \leq \epsilon$, for all $i = 1, \ldots, m$. Choosing $K_0, K_1, K_2, \ldots, K_m$ to be translates of the compact K such that $y_i \in K_i$ for all i, we observe, by the claim just proven, that $K_{i-1} \cap K_i \neq \emptyset$ for all $i = 1, \ldots, m$. By construction, whenever $gK \cap K \neq \emptyset$ we may express g in the form sh for $s \in \widehat{Q}$ and $h \in H_r$ (for some r). Thus the sequence $K_0, K_1, K_2, \ldots, K_m$ gives rise to a path of length at most $2m$ joining p to q in Γ. Thus $d_\Gamma(p, q) \leq 2m$. On the other hand $(m-1)\epsilon < \ell(\gamma) \leq d_X(f(p), f(q)) + \epsilon$. Combining these inequalities results in

$$d_\Gamma(p, q) \;<\; \frac{2}{\epsilon}d_X(f(p), f(q)) + 4\,.$$

This completes the proof that the map $f : \Gamma_0 \to X$ is a quasi-isometric embedding. Clearly, since the compact K is bounded, any point in X is a bounded distance from a point in the orbit of x_0, and so the map f is in fact a quasi-isometry. $\square$

5.1. **Complexes of groups.** A rather general construction is to describe a group G as the fundamental group of a complex of groups (see [5]). If G is the fundamental group of a finite complex $(Y, \mathcal{G})$ of groups which is developable and whose universal cover is Gromov hyperbolic, then G is weakly hyperbolic relative to $\mathcal{G}$. As discussed in the preceding section, each Artin group is the fundamental group of a finite complex of groups, leading to the results stated there.

5.2. **Mapping class groups.** A further example is that of the mapping class group $Mod(S)$ of a closed orientable surface S of higher genus. Recall that the complex of curves $\mathcal{C}(S)$ associated to the closed surface S is defined to be the simplicial complex whose vertices are the nontrivial isotopy classes of simple closed curves and whose simplices are spanned by sets of vertices which are simultaneously represented by mutually disjoint (non-parallel) simple closed curves. The group $Mod(S)$ acts naturally on this complex (with rather large vertex stabilizers). In [14], Masur and Minsky showed that the complex of curves $\mathcal{C}(S)$ is a Gromov hyperbolic space and used this to prove that the mapping class group $Mod(S)$ is weakly relatively hyperbolic with respect to subgroups $H_C := \{g \in Mod(S) : g[C] = [C]\}$ for a finite collection of simple closed curves C in S. Their proof passes through a modified version of Teichmuller space ("electric space") which they show to be quasi-isometric to $\mathcal{C}(S)$. A proof of this result may also be obtained by applying Theorem 5.1 directly to the complex $\mathcal{C}(S)$. This avoids introducing the action on Teichmuller space while still invoking the hyperbolicity of the curve complex as proved in [14].

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References

[1] **J W Anderson, J Aramonaya, K J Shackleton**, *A simple criterion for non-relative hyperbolicity and one-endednes of groups*, preprint, University of Southampton (2005), arXiv:math.GT/0504271
[2] **J Behrstock, C Drutu, L Mosher**, *Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity*, preprint (2005), arXiv:math.GT/0512592
[3] **N Bourbaki**, *Groupes et algèbres de Lie, Chapitres IV–VI*, Hermann, Paris (1968)
[4] **B H Bowditch**, *Relatively hyperbolic groups*, preprint, University of Southampton (1997)
[5] **M R Bridson, A Haefliger**, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag (1999)
[6] **R Charney**, *The Deligne complex for the four string braid group*, Trans. Amer. Math. Soc. 356 (2004), no. 10, 3881–3897
[7] **R Charney, M W Davis**, *The $K(\pi, 1)$-problem for hyperplane complements associated to infinite reflection groups*, J. Amer. Math. Soc. 8 (1995) 597–627
[8] **J Crisp**, *Automorphisms and abstract commensurators of 2-dimensional Artin groups*, Geometry and Topology 9 (2005) 1381–1441
[9] **B Farb**, *Relatively hyperbolic groups*, Geom. Funct. Anal. 8 (1998), 810-840
[10] **M Gromov**, *Hyperbolic Groups*, in "Essays in Group Theory (G.M.Gersten, editor), MSRI publ. 8, 1987, 75-263
[11] **M Gromov**, *Asymptotic invariants of infinite groups*, in "Geometric group theory, Vol. 2" (Sussex, 1991), 1295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993
[12] **I Kapovich, P Schupp**, *Relative hyperbolicity and Artin groups*, Geom. Dedicata 107 (2004), 153–167.
[13] **H van der Lek**, *The Homotopy Type of Complex Hyperplane Complements*, Ph.D. Thesis, University of Nijmegen (1983)
[14] **H A Masur, Y N Minsky**, *Geometry of the complex of curves I: hyperbolicity*, Invent. Math. 138 (1999), no. 1, 103–149
[15] **G Moussong**, *Hyperbolic Coxeter groups*, Ph.D. Thesis, Ohio State University (1988)

[16] **A Yaman**, *A topological characterisation of relatively hyperbolic groups*, J. Reine Angew. Math. 566 (2004), 41–89.

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